# MATH 320: PRACTICE PROBLEMS FOR THE FINAL

There will be eight problems on the final. The following are sample problems.

**Problem 1.** *Let $\mathcal{F}$ be the vector space of all real valued functions on the real line (i.e. $\mathcal{F} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$). Determine whether the following are subspaces of $\mathcal{F}$. Prove your answer.*

(1) $\{f \in \mathcal{F} \mid f(x) = -f(-x)$ *for all* $x\}$.
(2) $\{f \in \mathcal{F} \mid f(0) = 1\}$.
(3) $\{f \in \mathcal{F} \mid f(1) = 0\}$.

**Problem 2.** *Suppose that $T : V \to V$. Recall that a subspace $W$ is $T$-invariant if for all $x \in W$, we have that $T(x) \in W$.*

(1) *Prove that* $\operatorname{ran}(T)$, $ker(T)$ *are both $T$-invariant.*
(2) *Suppose that $W$ is a $T$-invariant subspace and $V = \operatorname{ran}(T) \oplus W$. Show that $W \subset ker(T)$.*

**Problem 3.** *Suppose that $T : V \to W$ is a linear transformation and $\{v_1, ..., v_n\}$ is a basis for $V$. Prove that $T$ is an isomorphism if and only if $\{T(v_1), ..., T(v_n)\}$ is a basis for $W$.*

**Problem 4.** *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be $T(\langle x_1, x_2, x_3\rangle) = \langle x_1 - x_2, x_2 - x_3, x_3 - x_1\rangle$. Let $\beta = \{\langle 1, 1, 1\rangle, \langle 1, 1, 0\rangle, \langle 1, -1, 0\rangle\}$ and let $e$ be the standard basis for $\mathbb{R}^3$.*

(1) *Find* $[T]_e$.
(2) *Find* $[T]_\beta$
(3) *Find an invertible matrix $Q$ such that* $[T]_\beta = Q^{-1}[T]_e Q$.

**Problem 5.** *Prove the theorem that a linear transformation is one-to-one if and only if it has a trivial kernel.*

**Problem 6.** *Determine if the following systems of linear equations are consistent*

(1)
$$\begin{aligned} &x + 2y + 3z = 1 \\ &x + y - z = 0 \\ &x + 2y + z = 3 \end{aligned}$$

(2)
$$\begin{aligned} &x + 2y - z = 1 \\ &2x + y + 2z = 3 \\ &x - 4y + 7z = 4 \end{aligned}$$

**Problem 7.** *Suppose that $A, B$ are two $n \times n$ matrices. Prove that the rank of $AB$ is less than or equal to the rank of $B$.*

**Problem 8.** *Suppose $A, B$ are $n \times n$ matrices, such that $B$ is obtained from $A$ by multiplying a row of $A$ by a nonzero scalar $c$.. Prove that* $\det(B) = c\det(A)$. *(You can use the definition of determinant by expansion along any row or column.)*

**Problem 9.** *Suppose $M$ is an $n \times n$ matrix that can be written in the form*

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

*where $A$ is a square matrix. Show that* $\det(M) = \det(A)$.

**Problem 10.** *A matrix $A$ is called nilpotent if for some positive integer $k$, $A^k = 0$. Prove that if $A$ is a nilpotent matrix, then $A$ is not invertible.*

**Problem 11.** *An $n \times n$ matrix $A$ is called orthogonal if $AA^t = I_n$. Prove that if $A$ is orthogonal, then $|\det A| = 1$.*

**Problem 12.** *Let $A$ be an $n \times n$ matrix. Prove that if $A$ is diagonalizable, then so is $A^t$.*

**Problem 13.** *Let*

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

*Show that $A$ is diagonalizable over $\mathbb{R}$ and find an invertible matrix $C$ such that $C^{-1}AC = D$ where $D$ is diagonal.*

**Problem 14.** *Let $T : V \to V$ be a linear transformation and let $x \in V$. Let $W$ be the $T$-cyclic subspace of $V$ generated by $x$. I.e. $W = Span(\{x, T(x), T^2(x), ...\})$.*

1. *Show that $W$ is $T$ - invariant.*
2. *Show that $W$ is the smallest $T$-invariant subspace containing $x$ (i.e. show that any $T$-invariant subspace that contains $x$, also contains $W$).*

**Problem 15.** *Let*

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

*Use the Cayley-Hamilton theorem to show that $A^2 - 2A + 5I$ is the zero matrix.*